Nonlinear Sensitivity Coefficients and Corrections in System Identification

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The evaluation of sensitivity coefficients, which are the changes in the eigencharacteristics with respect to changes in the structural parameters, is important in the identification of structural systems. In such studies, the commonly used linearized sensitivity coefficients for system identification have occasionally resulted in poor convergence or even divergence during the iteration process to update the analytical models for better correlation with the test data. Nonlinear sensitivity coefficients and nonlinear correction terms, usually eliminated during the linearization process, have been derived to determine sensitivity coefficients for linear systems. Their application to simulated example problems indicates a significant improvement in the convergence of the iteration processes to update the analytical model to correlate with test data. The successful application of combining the nonlinear sensitivity coefficients and nonlinear correction terms with the multiple boundary condition tests enhances the ability to validate large space structures by ground tests.

Introduction

In order to meet the mission objectives and to assure structural adequacy, most space programs require that the analytical model of the structure be updated using a systems identification process to correlate it with ground or in-flight test data. The change in the eigencharacteristics of an analytical model as a function of its structural parameters can easily be determined. However, establishing systematic modifications of the structural parameters that update the analytical model to improve its correlation with the test data has proved to be a difficult task. Many system identification procedures that have been successful on small computer-simulated problems have not been successful on realistic problems. With the apparent future need for large space structures, reliable system identification procedures are mandatory to verify the eigencharacteristics of the structure in its on-orbit configuration.

The most widely used approach in updating the analytical model is to adjust the various structural parameters in the model to minimize the difference between the test and analytical data using a least-squares approach. The success of this approach is directly dependent on the ability to evaluate the sensitivity coefficients of the eigencharacteristics accurately with respect to the various structural parameters selected as candidates for the change in the model. Numerous methods for calculating the required sensitivity coefficients have been published.[1-10] Most of the currently published methods utilize only linearized sensitivity coefficients because the nonlinear terms, which are assumed to be small, are neglected. The linearized sensitivity coefficients represent the largest slope from which the next iteration of the analytical model is determined. This approach has proved to be successful in optimal design. The utilization of linearized sensitivity coefficients for system identification, which update the analytical model to better correlate with the test data, has resulted in a large number of iterations. Occasionally, the solution has diverged or converged to unacceptable results, such as negative areas or moments of inertia of some structural elements.

The limitations of the linearized sensitivity coefficients lead to the examination of the linearizing approximations, the nonlinear terms of which were assumed to be small relative to the terms that were retained. The new nonlinear sensitivity method presented in this paper retains the nonlinear terms not accounted for in a linear approach.[1-9] The nonlinear sensitivity method provides a more rapid and stable convergence for the identification of system parameters that minimize the difference between the test and analytical results in the least-square sense. The proposed nonlinear sensitivity coefficients approach the linearized sensitivity coefficients when the analytical eigenvectors approach the test eigenvectors. To demonstrate the value of the proposed nonlinear sensitivity coefficients and nonlinear corrections, its application to a 6.096 m long beam modeled by 16 beam elements is presented. An analytical model with simulated errors of $\pm 30\%$ in the structural elements is used to establish the ability of the proposed method to detect errors in the analytical model when correlating the analytical data to the test data. The simulated test data are obtained by changing the cross section of the elements in the models. The linear least-squares method to detect the errors in the simply supported beam failed because the linearized sensitivity coefficients of the simply supported beam failed because the sensitivity coefficients matrix on the same problem resulted in the correct results in one iteration. The rapid convergence of the proposed method is also demonstrated for the same beam cantilevered from one end. Other examples are presented to illustrate the rate of convergence and the versatility of the proposed method.

Nonlinear Sensitivity Coefficients and Nonlinear Corrections for a Linear Dynamic System

The equation of motion in matrix form, without external forces, of an undamped dynamic system having a mass matrix of order $n, [M]$, and a stiffness matrix of order $n, [K]$, is

$$[M]\{\ddot{X}\} + [K]\{x\} = \{0\} \tag{1}$$

where $\{X\}$ is an n degree-of-freedom displacement vector, and $(\dot{\ })$ a derivative with respect to time, [ ] a matrix, and

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{ } a vector, and subscript ij the element in the matrix or the vector. The system has n eigencharacteristics (λ_i eigenvalues and ϕ_i eigenvectors, $i=1,...,n$). Both λ_i and ϕ_i are nonlinear functions of the elements of $[M]$ and $[K]$. In structural dynamics, Eq. (1) is usually rewritten in terms of eigencharacteristics as follows:

$$-[\lambda][M][\phi]+[K][\phi]=\{0\} \tag{2}$$

where $[\lambda]$ is a diagonal eigenvalue matrix and $[\phi]$ an $n\times n$ eigenvector matrix.

In the development of an analytical model, some degree of error in the formulation of either $[M]$ or $[K]$ is unavoidable; also, errors exist in the test data. Thus, discrepancies between the analytical model and the test data exist to varying degrees in all structures. In system identification, an analytical model representing the physical system and the first L eigencharacteristics, $L<n$, is assumed to be available. An incomplete set of test data are usually measured. Based upon this information, the elements of $[M]$ and $[K]$ of the given analytical model are perturbed to improve the correlation of the analytical results with the test data. The system identification is achieved by calculating the sensitivity coefficients of the parameters in both $[M]$ and $[K]$ and then using the least-squares method to minimize the difference between the analytical and test data. The iteration process may result in slow convergence to the correct results, convergence to physically unreasonable results, or even divergence.

For completeness, the linearized sensitivity coefficient formulations are briefly reviewed. The sensitivity coefficients of the eigenvalues obtained from the derivatives of the equations of motion are reported in the literature[1-9] as

$$\mathrm{d}\lambda_i=\sum_{\gamma=1}^{n}\sum_{s=1}^{n}\frac{\partial\lambda_i}{\partial K_{\gamma s}}\mathrm{d}K_{\gamma s}+\sum_{\gamma=1}^{n}\sum_{s=1}^{n}\frac{\partial\lambda_i}{\partial M_{\gamma s}}\mathrm{d}M_{\gamma s} \tag{3a}$$

$$\frac{\partial\lambda_i}{\partial K_{\gamma s}}=\phi_{\gamma i}\phi_{si} \tag{3b}$$

$$\frac{\partial\lambda_i}{\partial M_{\gamma s}}=-\lambda_i\phi_{\gamma i}\phi_{si} \tag{3c}$$

where the eigenvectors are normalized to unity with respect to the analytical mass matrix.

The sensitivity coefficients of the eigenvectors are more complicated than those for the eigenvalues. Of the many formulations reported in the literature, the one by Collins and Thomson[8] is

$$(\lambda_i-\lambda_j)\{\phi_j\}^T[M]\{\mathrm{d}\phi_i\}=\{\phi_j\}^T[\mathrm{d}K]\{\phi_i\}$$
$$-\lambda_i\{\phi_j\}^T[\mathrm{d}M]\{\phi_i\},\quad i\neq j \tag{4}$$

where $[\mathrm{d}M]$ and $[\mathrm{d}K]$ are the modifications to the analytical mass and stiffness matrices, respectively, and $\{\mathrm{d}\phi_i\}$ the difference of the eigenvectors between the test data and analytical results.

Consider the test data as a solution to Eq. (2) with the to-be-predicted mass matrix of order n, $[M]$ and the to-be-predicted stiffness matrix of order n, $[K]$. The relationship between the analytical and the to-be-predicted mass and stiffness matrices are

$$[M]=[M]+[\mathrm{d}M] \tag{5}$$

$$[K]=[K]+[\mathrm{d}K] \tag{6}$$

The measured eigencharacteristics are represented as

$$[\lambda]=[\lambda]+[\mathrm{d}\lambda] \tag{7}$$

$$[\phi]=[\phi]+[\mathrm{d}\phi] \tag{8}$$

where $[\mathrm{d}\lambda]$ and $[\mathrm{d}\phi]$ are the differences in the eigencharacteristics between the test data and the corresponding analytical results.

Substituting Eqs. (5-8) into Eq. (2), premultiplying by the transpose of a test eigenvector, and using the following orthogonality conditions of the analytical eigenvectors

$$[\phi]^T[M][\phi]=[I] \tag{9}$$

$$[\phi]^T[K][\phi]=[\lambda] \tag{10}$$

results in the following equations:

$$\mathrm{d}\lambda_i-\{\mathrm{d}\phi_i\}^T([K]-\lambda_i[M])\{\mathrm{d}\phi_i\}=A_{ij},\quad i=j \tag{11}$$

$$(\lambda_i-\lambda_j)\{\phi_j\}^T[M]\{\phi_i\}$$
$$-\{\mathrm{d}\phi_j\}^T([K]-\lambda_i[M])\{\mathrm{d}\phi_i\}=A_{ij},\quad i\neq j \tag{12}$$

where

$$A_{ij}=\{\phi_j\}^T[\mathrm{d}K]\{\phi_j\}+\{\phi_j\}^T[\mathrm{d}K]\{\mathrm{d}\phi_i\}$$
$$+\{\mathrm{d}\phi_j\}^T[\mathrm{d}K]\{\phi_i\}+\{\mathrm{d}\phi_j\}^T[\mathrm{d}K]\{\mathrm{d}\phi_i\}$$
$$-\lambda_i\{\phi_j\}^T[\mathrm{d}M]\{\phi_i\}-\lambda_i\{\phi_j\}^T[\mathrm{d}M]\{\mathrm{d}\phi_i\}$$
$$-\lambda_i\{\mathrm{d}\phi_j\}^T[\mathrm{d}M]\{\phi_i\}-\lambda_i\{\mathrm{d}\phi_j\}^T[\mathrm{d}M]\{\mathrm{d}\phi_i\}$$

and

$$\{\phi_j\}^T[M]\{\mathrm{d}\phi_j\}+\{\phi_j\}^T[\mathrm{d}M]\{\phi_i\}+\{\phi_j\}^T[\mathrm{d}M]\{\mathrm{d}\phi_i\}$$
$$+\{\mathrm{d}\phi_j\}^T[M]\{\mathrm{d}\phi_i\}+\{\mathrm{d}\phi_j\}^T[M]\{\phi_i\}$$
$$+\{\mathrm{d}\phi_j\}^T[\mathrm{d}M]\{\phi_i\}+\{\mathrm{d}\phi_j\}^T[\mathrm{d}M]\{\mathrm{d}\phi_i\}=0 \tag{13}$$

The equations relate the changes in the mass matrix $[\mathrm{d}M]$ and stiffness matrix $[\mathrm{d}K]$ to the differences of the eigenvectors and eigenvalues between the test data and analytical results. The form of Eq. (13) assumes that the measured eigenvectors are normalized to unity. Expansion of the matrix Eqs. (11) and (12) results in

$$\mathrm{d}\lambda_i-\{\mathrm{d}\phi_i\}^T([K]-\lambda_i[M])\{\mathrm{d}\phi_i\}=\sum_{\gamma=1}^{n}\sum_{s=1}^{n}B_{\gamma s}^{ij}\mathrm{d}M_{\gamma s}$$
$$+\sum_{\gamma=1}^{n}\sum_{s=1}^{n}C_{\gamma s}^{ij}\mathrm{d}K_{\gamma s},\quad i=j \tag{14}$$

$$(\lambda_i-\lambda_j)\{\phi_i\}^T[M]\{\mathrm{d}\phi_i\}-\{\mathrm{d}\phi_j\}^T([K]-\lambda_i[M])\{\mathrm{d}\phi_i\}$$
$$=\sum_{\gamma=1}^{n}\sum_{s=1}^{n}B_{\gamma s}^{ij}\mathrm{d}M_{\gamma s}+\sum_{\gamma=1}^{n}\sum_{s=1}^{n}C_{\gamma s}^{ij}\mathrm{d}K_{\gamma s},\quad i\neq j \tag{15}$$

where

$$B_{\gamma s}^{ij}=\phi_{\gamma j}\phi_{si}+\phi_{\gamma j}\mathrm{d}\phi_{si}+\mathrm{d}\phi_{\gamma j}\phi_{si}+\mathrm{d}\phi_{\gamma j}\mathrm{d}\phi_{si} \tag{16}$$

$$C_{\gamma s}^{ij}=-\lambda_i B_{\gamma s}^{ij} \tag{17}$$

Equations (14) and (15) indicate the change in eigencharacteristics as the structural parameters are varied. Equations (16) and (17) are the sensitivity coefficients of the system. They are similar to the linearized sensitivity coefficients except that the higher-order terms, which include differences of the eigenvectors between the test data and the analytical results, are retained. The last three terms on the right-hand side of Eqs. (16) and (17) are neglected in the linearized sensitivity derivation. Equations (16) and (17) are referred to as the *nonlinear sensitivity coefficients* of the system. The second term on the left-hand side of Eqs. (14) and (15) is a scalar quantity referred to as a *nonlinear correction* term. These terms are assumed to be a higher-order negligible term because they represent the difference of the eigenvectors in the linearized sensitivity formulations. This assumption will be re-examined by evaluating the magnitude of the terms in the paper. The terms are zero if the difference between the analytical and test eigenvectors is exactly proportional to the analytical eigenvectors. Also, the term can be erroneously large if either the analytical or the test eigenvectors are not properly normalized. The initial evaluation would suggest the *nonlinear correction* terms are small because in many finite-element analysis, the order of magnitude of the eigenmatrix is 10^6, whereas the difference in the eigenvectors is on the order of 0.01, as shown in Table 1. However, the *nonlinear correction* can be significant because it is a sum of a large number of terms. The evaluation of the *nonlinear correction* for the sample problem of Fig. 1 as shown in Tables 2 and 3 indicates that they are not negligible, but of the same magnitude as the significant terms. When all the higher-order and nonlinear terms are neglected, Eqs. (16) and (17) reduce to the linearized sensitivity coefficients of Eqs. (3) and (4).

System Identification by the Least-Squares Method

In structural dynamic testing, only a very small subset of the theoretically available eigenvalues and eigenvectors can be accurately measured. Consequently, in the system identification process when the least-squares method is applied, the number of equations for the identification are less than the unknown parameters and hence it is an undetermined system. Since aerospace structures are usually designed using many members with identical geometric and material properties, the indentical members can be grouped together and considered as one variable in the system identification process.[11,12] Thus,

$$\begin{bmatrix} \dfrac{\partial \lambda_i}{\partial r} \\ \\ \dfrac{\partial \phi_i}{\partial r} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial \lambda_i}{\partial K} & \dfrac{\partial \lambda_i}{\partial M} \\ \\ \dfrac{\partial \phi_i}{\partial K} & \dfrac{\partial \phi_i}{\partial M} \end{bmatrix} \begin{bmatrix} \dfrac{\partial K}{\partial r} \\ \\ \dfrac{\partial M}{\partial r} \end{bmatrix} \tag{18}$$

where r can be one scalar representing a large group of elements to be modified. Whenever the grouping of similar structures or engineering judgment does not reduce the unknown parameters to less than equal of the number of equations, additional test data must be obtained. The multiple boundary condition test (MBCT) method is another test approach that can provide a substantially larger number of accurate test data.[13-15] The successful application of combining the MBCT approach, which allows for a larger number of accurate test results, with the retention of the nonlinear terms as proposed in this paper is illustrated by numerical examples.

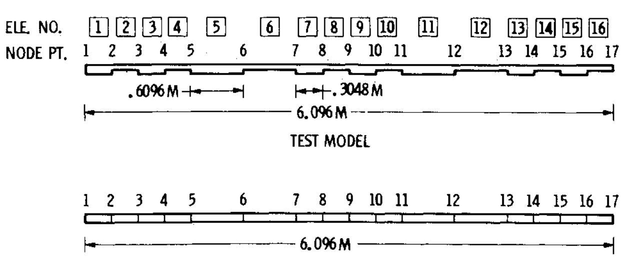

Fig. 1 Test and analytical models of a beam.

Applications

The quantity of reliable test data is always less than the degrees of freedom or parameters of the analytical model. If eigenvectors are measured, the degree of freedom of the eigenvectors not directly measured can, in general, be generated by using the analytical results as a baseline to interpolate or extrapolate to obtain the displacements at these degrees of freedom. As indicated by Eqs. (14-17), the nonlinear corrections and nonlinear sensitivity coefficients are highly dependent upon the eigenvectors. As the nonlinear sensitivity coefficients and the nonlinear corrections are derived from the equations of motion without any truncations, it is a closed-form solution between the up-dated system matrices and the differences between the test and analytical eigencharacteristics. Note that, if the effect of the eigenvectors is neglected, the nonlinear case reduces to the linearized sensitivity coefficient. In this study, the affect of measurement errors on the rate and stability of convergence is not evaluated.

Simulation Examples

To compare the rate and stability of convergence and the accuracy of the nonlinear sensitivity coefficients to the linearized sensitivity coefficients, a 6.096 m beam modeled by 16 three-dimensional elements, 0.3048 or 0.6096 m long, as shown in Fig. 1, is used for the numerical simulation. The moment of inertia of the elements alternatively changes between 15.466×10^{-6} and 8.359×10^{-6} m^4 for the odd- and even-numbered beam elements, respectively. The boundary conditions are simply supported in the first case and cantilevered at the left side in the second case. In the computer simulation, the unperturbed model is selected to represent the analytical model and the perturbed model the test hard-

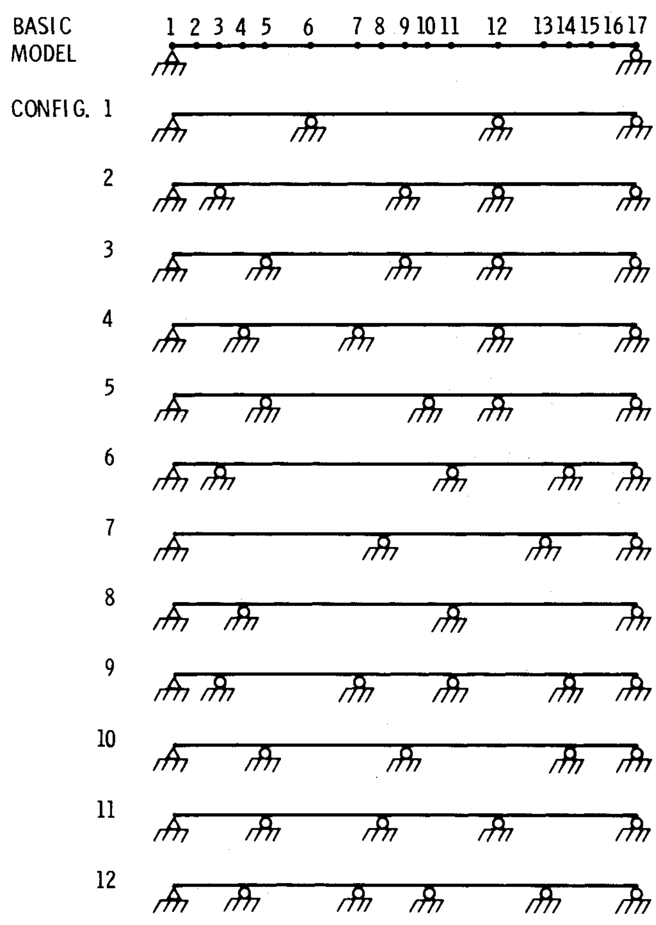

Fig. 2 MBCT test configurations.

Table 1 Eigencharacteristics of test and analytical models

Mode	Model[a]	Eigen-value, Hz	2	3	4	5	6	7	8	9
1	A	26.678	0.479	0.942	1.384	1.792	2.467	2.901	3.013	3.051
	T	25.449	0.475	0.938	1.379	1.784	2.472	2.913	3.019	3.051
2	A	107.899	0.958	1.822	2.504	2.937	2.927	1.805	0.949	0.000
	T	102.898	0.936	1.769	2.431	2.844	2.895	1.865	1.009	0.007
3	A	235.177	−1.370	−2.444	−2.988	−2.880	−0.971	1.693	2.592	2.914
	T	224.623	−1.439	−2.538	−3.097	−2.960	−0.999	1.630	2.563	2.914
4	A	423.247	1.726	2.788	2.770	1.656	−1.917	−2.956	−1.834	0.000
	T	403.886	1.795	2.849	2.819	1.688	−2.091	−2.964	−1.664	0.185
5	A	664.489	2.049	2.891	2.017	−0.094	−3.285	−0.102	2.004	2.869
	T	634.253	1.863	2.571	1.768	−2.229	−3.084	−0.161	2.016	2.851
6	A	956.854	−2.275	−2.678	−0.877	1.644	2.138	−2.955	−2.637	0.000
	T	915.274	−2.432	−2.784	−0.812	1.603	2.301	−2.818	−2.812	−0.440

[a]T = test model, A = analytical model.

Mode	Model[a]	Eigen-value, Hz	10	11	12	13	14	15	16
1	A	26.678	3.013	2.901	2.467	1.792	1.384	0.942	0.477
	T	25.449	3.007	2.889	2.462	1.799	1.389	0.946	0.478
2	A	107.899	−0.949	−1.805	−2.927	−2.937	−2.504	−1.822	−0.958
	T	102.898	−0.889	−1.744	−2.956	−3.028	−2.574	−1.873	−0.980
3	A	235.177	2.592	1.693	−0.971	−2.880	−2.988	−2.444	−1.370
	T	224.623	2.620	1.749	−0.940	−2.793	−2.879	−2.352	−1.304
4	A	423.247	1.834	2.956	1.917	−1.656	−2.770	−2.788	−1.726
	T	403.886	1.999	2.943	1.738	−1.618	−2.712	−2.770	−1.654
5	A	664.489	2.003	−0.102	−3.285	−0.094	2.017	2.891	2.049
	T	634.253	1.966	−0.035	−3.462	−0.177	2.269	3.208	2.223
6	A	956.854	2.637	2.955	−2.138	−1.644	0.876	2.678	2.275
	T	915.274	2.502	3.081	−1.963	−1.662	0.903	2.530	2094

[a]T = test model, A = analytical model.

Table 2 Nonlinear correction terms of simply supported beam

Mode	1	2	3	4	5	6
1	2,529	105	22	−18,251	946	−35,715
2	105	41,561	41,849	1,288	26,909	−3,687
3	22	41,849	192,550	−43,976	7,691	338,810
4	−18,251	1,288	−43,976	633,060	446,650	−78,756
5	946	26,909	7,691	446,650	1,554,000	1,000,100
6	−35,715	−3,687	338,810	−78,756	1,000,100	3,107,000

Table 3 Nonlinear corrections and nonlinear sensitivities of simply supported beam

	Eigenvalues				Linear sensitivity		Nonlinear sensitivity	
Mode	Anal.	Test	(Test − Anal.)	Nonlinear correction	Odd no. elements	Even no. elements	Odd no. elements	Even no. elements
1	28,097	25,568	−2,529	2,529	492	492	241	831
2	459,613	418,002	−41,611	41,561	8,040	8,040	3,917	13,617
3	2,183,493	1,991,913	−191,580	192,550	38,196	38,196	19,165	63,962
4	7,072,100	6,439,880	−632,220	633,060	123,710	123,710	61,008	208,564
5	17,431,550	15,881,290	−1,550,260	1,554,000	304,925	304,926	151,102	513,118
6	36,145,240	33,072,120	−3,073,120	3,107,000	632,279	632,278	326,270	1,046,990

Table 4 Nonlinear corrections and nonlinear sensitivities of cantilevered beam for first iteration

	Eigenvalue, rad/s				Linearized sensitivity coef.		Nonlinear sensitivity coef.	
Mode	Anal.	Test	Test – Anal.	Nonlinear correction	Odd no. element	Even no. element	Odd no. element	Even no. element
1	3,537	3,355	– 182	340	70	53	37	97
2	137,522	129,144	– 8,378	12,874	2,644	2,167	1,392	3,870
3	1,125,558	1,046,859	– 78,699	106,650	21,190	18,100	10,747	32,362
4	4,062,632	3,818,266	– 244,366	361,890	77,136	64,996	42,046	112,747
5	10,941,510	10,345,300	– 596,210	940,260	209,703	173,090	118,255	297,434
6	24,044,990	22,362,890	– 1,682,100	2,192,500	447,303	393,920	233,043	684,885

Table 5 First iteration results of simply supported beam

Without nonlinear corr.	With nonlinear corr.	Linearized sensitivity coef.	Nonlinear sensitivity coef.	I_1 $10^{-6}\times m^4$	I_2
x		x		Divergent	
	x	x		Divergent	
x			x	13.984	10.027
	x		x	15.454	8.332

Table 6 Results of iteration for cantilevered beam

Without nonlinear corr.	With nonlinear corr.	Linearized sensitivity coef.	Nonlinear sensitivity coef.	First iteration I_1	I_2	Second iteration I_1 $10^{-6}\times m^4$	I_2	Third iteration I_1	I_2
x		x		16.378	5.017	14.513	7.840	15.466	8.328
	x	x		15.706	8.111	14.255	7.346	15.466	8.328
x			x	15.187	9.756	15.885	8.936	15.446	8.328
	x		x	15.482	8.323	15.466	8.328	15.466	8.328

Table 7 Nonlinear corrections and nonlinear sensitivities of cantilevered beam for third iteration

	Eigenvalue, rad/s				Linearized sensitivity coef.		Nonlinear sensitivity coef.	
Mode	Anal.	Test	Test – Anal.	Nonlinear correction	Odd no. element	Even no. element	Odd no. element	Even no. element
1	3,154	3,355	– 200	0.01	37	98	37	98
2	121,415	129,144	– 7,728	0.40	1,397	3,859	1,392	3,870
3	984,270	1,046,859	– 62,589	1.60	10,790	32,283	10,747	32,362
4	3,589,669	3,818,266	– 228,599	8.50	42,196	112,470	42,047	112,747
5	9,725,450	10,345,300	– 619,850	24.40	118,641	296,717	118,254	297,434
6	21,025,500	22,362,890	– 1,337,390	53.00	233,931	683,236	233,043	684,885

ware. The test data are assumed to contain no measurement noise.

Equations (14) and (15) show the same degree of correction by including nonlinear terms in either the mass or stiffness sensitivity coefficients. In the illustrations, for simplicity without loss of generality, only the stiffness terms are changed. The coefficients relating to the change in frequency with respect to a change in the moment of inertia [in Eq. (18), r = the moment of inertia] is calculated by Eq. (14). Using the difference between the analytical and test eigenvectors $d\phi$ and the parameters of the analytical model, the change in frequency $d\lambda$ is determined using the evaluation of both the nonlinear sensitivity coefficients and nonlinear corrections. The analytical model is identical to the one depicted in Fig. 1, except that the moment of inertia in the analytical model is assumed to be 11.897×10^{-6} m^4, an error of $\pm30\%$ in the stiffness matrix. The masses of the elements are unchanged. The results for case 1 for the lowest six modes are shown in Table 1.

The linearized sensitivity coefficients, nonlinear sensitivity coefficients, and the nonlinear corrections terms are calculated using MSC/NASTRAN and are listed in Tables 3 and 4 for the simply supported beam and the cantilevered beam, respectively. The large nonlinear corrections shown in Tables 2–4 invalidated the assumption used to linearize the equations.

System indentification using the linearized sensitivity coefficients of the simply supported beam failed because the sensitivity coefficients matrix on the left-hand side of the least-squares method is singular. By introducing the nonlinear modifications to the sensitivity coefficients and/or adding the nonlinear corrections, the analytical model converged to the correct results in one iteration, as noted in Table 5.

For the cantilevered beam, three iterations were required to converge to the correct results using the linearized sensitivity coefficients, whereas convergence to the correct results is achieved in one iteration by using the nonlinear sensitivity coefficients. The results of various iterations using both the linear and nonlinear sensitivity approaches are listed in Table 6. In the third iteration (Table 7), the contributions of the nonlinear terms are insignificant because the difference between the initial estimate of the eigenvector and

Table 8 Moment of interia of every element of simply supported beam by MBCT method, moderate deviations

Elem. no.	Analytical I $10^{-8} \times m^4$	Test I	Diff., %	Linearized sensitivity First iter.	Second iter.	I identified by MBCT method, one iteration (configurations × modes), $10^{-8} \times m^4$ 12×3	12×7	8×7	4×7
1	3.46847	3.81851	10	4.62475	−27.69479	3.82059	3.81851	3.81851	3.81851
2	3.46847	3.64494	5	3.51591	5.84680	3.64452	3.64494	3.64452	3.64452
3	3.46847	3.47137	0	3.25285	6.50486	3.47095	3.47095	3.47095	3.47137
4	3.46847	3.29780	−5	3.08094	3.20873	3.29697	3.29780	3.29780	3.29780
5	3.46847	3.12423	−10	3.17168	3.30571	3.12423	3.12423	3.12423	3.12423
6	3.46847	2.77710	−20	2.92652	1.62830	2.77710	2.77710	2.77710	2.77710
7	3.46847	3.12423	−10	2.88823	3.40977	3.12423	3.12423	3.12423	3.12423
8	3.46847	3.29780	−5	3.66866	0.58938	3.29780	3.29780	3.29780	3.29739
9	3.46847	3.47137	0	3.04556	5.11215	3.47137	3.47137	3.47137	3.47095
10	3.46847	3.64494	5	4.02038	3.12673	3.64452	3.64494	3.64494	3.64452
11	3.46847	3.81851	10	3.40810	3.69988	3.81809	3.81809	3.81851	3.81892
12	3.46847	4.16564	20	4.07865	4.38042	4.16606	4.16564	4.16564	4.16564
13	3.46847	3.81851	10	3.35483	6.05990	3.81892	3.81851	3.81851	3.81976
14	3.46847	3.64494	5	4.81663	1.55879	3.64411	3.64494	3.64494	3.64202
15	3.46847	3.47137	0	1.17752	1.86097	3.47179	3.47137	3.47137	3.47137
16	3.46847	3.64494	5	4.22517	11.44300	3.64119	3.64494	3.64452	3.63287

the final results is small. This implies that the linearized sensitivity coefficients are adequate to estimate changes in the vicinity of the analytically estimated parameters. This observation helps clarify the success of linearized sensitivity coefficients in optimization solutions. When the linearized sensitivity techniques are used for system identification, the additional test data, from which the difference in the analytical and test eigenvectors can be evaluated, are neglected. The neglect of these data is one of the contributors to slow convergence or divergence to the incorrect results. The beam shown in Fig. 1 with 16 different moments of inertia, one for each beam element, is used for a numerical simulation to test the MBCT test concept proposed for validation of structures. Twelve different configurations, using the various boundary conditions shown in Fig. 2, are used to obtain the test data. The test data and analytical results of each configuration are generated. The test data are generated by varying the moment of inertia from −20 to +20% to simulate the errors in the analytical model. The individual elements before and after the changes are listed in Table 8. By using 12 configurations and 3 eigenvectors with the linearized sensitivity formulation, the moment of inertia of the first element becomes negative after the second iteration. Using the nonlinear corrections and sensitivities, convergence to the correct moment of inertia is achieved in one iteration and is insensitive to the number of configurations and eigenvectors, as long as the product of the number of configurations and the number of eigenvectors provides more equations than the unknowns.

Conclusion

The retention of the nonlinear higher-order terms (which are usually neglected) has improved the capability to modify the analytical model to better correlate it with its test data. Retention of the *nonlinear sensitivity coefficients* and *nonlinear corrections* in the system identification helps to assure a more rapid convergence, as well as convergence to the desired results. Numerical examples illustrate that the linearizing approximations frequently made are not valid for all problems. The retention of nonlinear terms for system identification as presented in this paper and the concept of the multiple boundary condition tests (MBCT) can update the unknowns in the model correctly and rapidly.

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